

WHEN RIGHT n -ENGEL ELEMENTS OF A GROUP FORM A SUBGROUP?

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ABSTRACT. Let $R_n(G)$ denotes the set of all right n -Engel elements of a group G . We show that in any group G whose 5th term of lower central series has no element of order 2, $R_3(G)$ is a subgroup. Furthermore we prove that $R_4(G)$ is a subgroup for locally nilpotent groups G without elements of orders 2, 3 or 5; and in this case the normal closure $\langle x \rangle^G$ is nilpotent of class at most 7 for each $x \in R_4(G)$. Using a group constructed by Newman and Nickel we also show that, for each $n \geq 5$, there exists a nilpotent group of class $n+2$ containing a right n -Engel element x and an element $a \in G$ such that both $[x^{-1},_n a]$ and $[x^k,_{n+1} a]$ are of infinite order for all integers $k \geq 2$. We finish the paper by proving that at least one of the following happens: (1) There is an infinite finitely generated k -Engel group of exponent n for some positive integer k and some 2-power number n . (2) There is a group generated by finitely many bounded left Engel elements which is not an Engel group.

1. Introduction and Results

Let G be any group and n a non-negative integer. For any two elements a and b of G , we define inductively $[a,_{n+1} b]$ the n -Engel commutator of the pair (a, b) , as follows:

$$[a,_{n+1} b] := a, \quad [a, b] := [a,_{1+1} b] := a^{-1}b^{-1}ab \text{ and } [a,_{n+1} b] = [[a,_{n+1} b], b] \text{ for all } n > 0.$$

An element x of G is called right (left, resp.) n -Engel if $[x,_{n+1} g] = 1$ ($[g,_{n+1} x] = 1$, resp.) for all $g \in G$. We denote by $R_n(G)$ ($L_n(G)$, resp.), the set of all right (left, resp.) n -Engel elements of G . A group G is called n -Engel if $G = L_n(G)$ or equivalently $G = R_n(G)$. It is clear that $R_0(G) = 1$, $R_1(G) = Z(G)$ the center of G , and W.P. Kappe [7] (implicitly) proved $R_2(G)$ is a characteristic subgroup of G . L.C. Kappe and Ratchford [8] have shown that $R_n(G)$ is a subgroup of G whenever G is a metabelian group, or G is a center-by-metabelian group such that $[\gamma_k(G), \gamma_j(G)] = 1$ for some $k, j \geq 2$ with $k + j - 2 \leq n$ and $n \geq 3$. Macdonald [9] has shown that the inverse or square of a right 3-Engel element need not be right 3-Engel. Nickel [15] generalized Macdonald's result to all $n \geq 3$. In fact he constructed a group with a right n -Engel element a neither a^{-1} nor a^2 is a right n -Engel element. The construction of Nickel's example was guided by computer experiments and arguments based on commutator calculus. Although Macdonald's example shows that $R_3(G)$ is not in general a subgroup of G , Heineken [5] has already shown that if A is the subset of a group G consisting of all elements a such that $a^{\pm 1} \in R_3(G)$, then A is a subgroup if either G has no element of order 2 or A consists only of elements having finite odd order. Newell [13] proved that the

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normal closure of every right 3-Engel element is nilpotent of class at most 3. In Section 2 we prove that if G is a $2'$ -group, then $R_3(G)$ is a subgroup of G . Nickel's example shows that the set of right 4-Engel elements is not a subgroup in general (see also first Example in Section 4 of [1]). In Section 3, we prove that if G is a locally nilpotent $\{2, 3, 5\}'$ -group, then $R_4(G)$ is a subgroup of G .

Traustason [17] proved that any locally nilpotent 4-Engel group H is Fitting of degree at most 4. This means that the normal closure of every element of H is nilpotent of class at most 4. More precisely he proved that if H has no element of order 2 or 5, then H has Fitting degree at most 3. Now by a result of Havas and Vaughan-Lee [4], one knows any 4-Engel group is locally nilpotent and so Traustason's result is true for all 4-Engel groups. In Section 3, by another result of Traustason [18] we show that the normal closure of every right 4-Engel element in a locally nilpotent $\{2, 3, 5\}'$ -group, is nilpotent of class at most 7.

Newman and Nickel [12] have shown that for every $n \geq 5$ there exists a nilpotent group G of class $n+2$ containing a right n -Engel element a and an element b such that $[b, n a]$ has infinite order. As we mentioned above, Nickel [15] has shown that for every $n \geq 3$ there exists a nilpotent group of class $n+2$ having a right n -Engel element a and an element b such that $[a^{-1}, n b] = [a^2, n b] \neq 1$. We have checked that the latter element in Nickel's example is of finite order whenever $n \in \{5, 6, 7, 8\}$. In Section 4, using the group constructed by Newman and Nickel we show that there exists a nilpotent group G of class $n+2$ such that $x \in R_n(G)$ and both $[x^{-1}, n a]$ and $[x^k, n a]$ have infinite order for every integer $k \geq 2$.

In [1] the following question has been proposed:

Question 1.1. *Let n be a positive integer. Is there a set of prime numbers π_n depending only on n such that the set of right n -Engel elements in any nilpotent or finite π'_n -group forms a subgroup?*

In Section 4 we negatively answers Question 1.1.

As far as we know there is no published example of a group whose set of (bounded) right Engel elements do not form a subgroup. But for the set of bounded left Engel elements there are some evidences supporting this idea that the subgroup-ness of bounded left Engel elements of a an arbitrary group should be false. We finish the paper by proving that at least one of the following happens:

- (1) There is an infinite finitely generated k -Engel group of exponent n for some positive integer k and some 2-power number n .
- (2) There is a group generated by finitely many bounded left Engel elements which is not an Engel group, where by an Engel group we mean a group in which for every two elements x and y , there exists an integer $k = k(x, y) \geq 0$ such that $[x, k y] = 1$.

Throughout the paper we have frequently use GAP `nq` package of Werner Nickel. All given timings were obtained on an Intel Pentium 4-1.70GHz processor with 512 MB running Red Hat Enterprise Linux 5.

2. Right 3-Engel elements

Throughout for any positive integer k and any group H , $\gamma_k(H)$ denotes the k th term of the lower central series of H . The main result of this section implies that $R_3(G)$ is a subgroup of G whenever G is a $2'$ -group. Newell [13] proved that

Theorem 2.1. *Let $G = \langle a, b, c \rangle$ be a group such that $a, b \in R_3(G)$. Then*

- (1) $\langle a, c \rangle$ is nilpotent of class at most 5 and $\gamma_5(\langle a, c \rangle)$ has exponent 2.
- (2) G is nilpotent of class at most 6.
- (3) $\frac{\gamma_5(G)}{\gamma_6(G)}$ has exponent 10. Furthermore $[a, c, b, c, c]^2 \in \gamma_6(G)$.
- (4) $\gamma_6(G)$ has exponent 2.

Theorem 2.2. *Let G be a group such that $\gamma_5(G)$ has no element of order 2. Then $R_3(G)$ is a subgroup of G .*

Proof. Let $a, b \in R_3(G)$ and $c \in G$. We first show that $a^{-1} \in R_3(G)$. We have

$$\begin{aligned} [a^{-1}, c, c, c] &= [[[a, c, a^{-1}]^{-1}[a, c]^{-1}, c], c] \\ &= [[a, c, a, a^{-1}, c][a, c, a, c], c][[a, c, c, [a, c]^{-1}][a, c, c]^{-1}, c] \\ &= [a, c, a, c, c][a, c, c, c]^{[a, c, c]} \\ &= [a, c, a, c, c] \end{aligned}$$

Therefore by Theorem 2.1 (2), $a^{-1} \in R_3(G)$. On the other hand

$$\begin{aligned} [ab, c, c, c] &= [[a, c][a, c, b][b, c], c, c] \\ &= [[a, c, c][a, c, c, [a, c, b]][[a, c, c], [b, c]][a, c, b, c][b, c, b, c, [b, c]][b, c, c], c] \\ &= [a, c, c, [b, c], c][a, c, b, c, c]. \end{aligned}$$

Now by Theorem 2.1 $[a, c, c, [b, c], c], [a, c, b, c, c]^2 \in \gamma_6(G)$ and thus $ab \in R_3(G)$. \square

Now we give a proof of Theorem 2.2 by using GAP **nq** package of Werner Nickel.

Second Proof of Theorem 2.2. By Theorem 2.1, we know that $\langle x, y, z \rangle$ is nilpotent if $x, y \in R_3(G)$ and $z \in G$. We now construct the largest nilpotent group $H = \langle a, b, c \rangle$ such that $a, b \in R_3(H)$ and $c \in H$, by **nq** package.

```
LoadPackage("nq"); #nq package of Werner Nickel #
F:=FreeGroup(4);a1:=F.1; b1:=F.2; c1:=F.3; x:=F.4;
L:=F/[LeftNormedComm([a1,x,x,x]),LeftNormedComm([b1,x,x,x])];
H:=NilpotentQuotient(L,[x]);
a:=H.1; b:=H.2; c:=H.3; d:=LeftNormedComm([a^{-1},c,c,c]);
e:=LeftNormedComm([a*b,c,c,c]); Order(d); Order(e);
C:=LowerCentralSeries(H); d in C[5]; e in C[5];
```

Then if we consider the elements $d = [a^{-1}, c, c, c]$ and $e = [ab, c, c, c]$ of H , we can see by above command in GAP that d and e are elements of $\gamma_5(H)$ and have orders 2 and 4, respectively. So, in the group G , we have $d = e = 1$. This completes the proof. \square

Note that, the second proof of Theorem 2.2 also shows the necessity of assuming that $\gamma_5(G)$ has no element of order 2.

3. Right 4-Engel elements

Our main result in this section is to prove the following.

Theorem 3.1. *Let G be a $\{2, 3, 5\}'$ -group such that $\langle a, b, x \rangle$ is nilpotent for all $a, b \in R_4(G)$ and any $x \in G$. Then $R_4(G)$ is a subgroup of G .*

Proof. Consider the ‘freest’ group, denoted by U , generated by two elements u, v with u a right 4-Engel element. We mean this by the group U given by the presentation

$$\langle u, v \mid [u, v] = 1 \text{ for all words } x \in F_2 \rangle,$$

where F_2 is the free group generated by u and v . We do not know whether U is nilpotent or not. Using the **nq** package shows that the group U has a largest nilpotent quotient M with class 8. By the following code, the group M generated by a right 4-Engel element a and an arbitrary element c is constructed. We then see that the element $[a^{-1}, c, c, c, c]$ of M is of order $375 = 3 \times 5^3$. Therefore the inverse of a right 4-Engel element of G is again a right 4-Engel element. The following code in GAP gives a proof of the latter claim. The computation was completed in about 24 seconds.

```

F:=FreeGroup(3); a1:=F.1; b1:=F.2; x:=F.3;
U:=F/[LeftNormedComm([a1,x,x,x,x])];
M:=NilpotentQuotient(U,[x]);
a:=M.1; c:=M.2;
h:=LeftNormedComm([a^-1,c,c,c,c]);
Order(h);

```

We now show that the product of every two right 4-Engel elements in G is a right 4-Engel element. Let $a, b \in R_4(G)$ and $c \in G$. Then we claim that

$$H = \langle a, b, c \rangle \text{ is nilpotent of class at most 7. } (*)$$

By induction on the nilpotency class of H , we may assume that H is nilpotent of class at most 8. Now we construct the largest nilpotent group $K = \langle a_1, b_1, c_1 \rangle$ of class 8 such that $a_1, b_1 \in R_4(K)$.

```

F:=FreeGroup(4);A:=F.1; B:=F.2; C:=F.3; x:=F.4;
W:=F/[LeftNormedComm([A,x,x,x,x]),LeftNormedComm([B,x,x,x,x])];
K:=NilpotentQuotient(W,[x],8);
LowerCentralSeries(K);

```

The computation took about 22.7 hours. We see that $\gamma_8(K)$ has exponent 60. Therefore, as H is a $\{2, 3, 5\}'$ -group, we have $\gamma_8(H) = 1$ and this completes the proof of our claim (*).

Therefore we have proved that any nilpotent group without elements of orders 2, 3 or 5 which is generated by three elements two of which are right 4-Engel, is nilpotent of class at most 7.

Now we construct, by the **nq** package, the largest nilpotent group S of class 7 generated by two right 4-Engel elements s, t and an arbitrary element g . Then one can find by GAP that the order of $[st, g, g, g]$ in S is 300. Since H is a quotient of S , we have that $[ab, c, c, c, c]$ is of order dividing 300 and so it is trivial, since H is a $\{2, 3, 5\}'$ -group. This completes the proof. \square

Corollary 3.2. *Let G be a $\{2, 3, 5\}'$ -group such that $\langle a, b, x \rangle$ is nilpotent for all $a, b \in R_4(G)$ and for any $x \in G$. Then $R_4(G)$ is a nilpotent group of class at most 7. In particular, the normal closure of every right 4-Engel element of group G is nilpotent of class at most 7.*

Proof. By Theorem 3.1, $R_4(G)$ is a subgroup of G and so it is a 4-Engel group. In [18] it is shown that every locally nilpotent 4-Engel $\{2, 3, 5\}'$ -group is nilpotent of class at most 7. Therefore $R_4(G)$ is nilpotent of class at most 7. Since $R_4(G)$ is a normal set, the second part follows easily. \square

Therefore, to prove that the normal closure of any right 4-Engel element of a $\{2, 3, 5\}'$ -group G is nilpotent, it is enough to show that $\langle a, b, x \rangle$ is nilpotent for all $a, b \in R_4(G)$ and for any $x \in G$. It may be surprising that Newell [13] has had a similar obstacle to prove that the normal closure of a right 3-Engel element is nilpotent in any group.

Corollary 3.3. *In any $\{2, 3, 5\}'$ -group, the normal closure of any right 4-Engel element is nilpotent if and only if every 3-generator subgroup in which two of the generators can be chosen to be right 4-Engel, is nilpotent.*

Proof. By Corollary 3.2, it is enough to show that a $\{2, 3, 5\}'$ -group $H = \langle a, b, x \rangle$ is nilpotent whenever $a, b \in R_4(H)$, $x \in H$ and both $\langle a \rangle^H$ and $\langle b \rangle^H$ are nilpotent. Consider the subgroup $K = \langle a \rangle^H \langle b \rangle^H$ which is nilpotent by Fitting's theorem. Now we prove that K is finitely generated. We have $K = \langle a, b \rangle^{\langle x \rangle}$ and since a and b are both right 4-Engel, it is well-known that

$$\langle a \rangle^{\langle x \rangle} = \langle a, a^x, a^{x^2}, a^{x^3} \rangle \text{ and } \langle b \rangle^{\langle x \rangle} = \langle b, b^x, b^{x^2}, b^{x^3} \rangle,$$

and so

$$K = \langle a, a^x, a^{x^2}, a^{x^3}, b, b^x, b^{x^2}, b^{x^3} \rangle.$$

It follows that H satisfies maximal condition on its subgroups as it is (finitely generated nilpotent)-by-cyclic. Now by a famous result of Baer [2] we have that a and b lie in the $(m + 1)$ th term $\zeta_m(H)$ of the upper central series of H for some positive integer m . Hence $H/\zeta_m(H)$ is cyclic and so H is nilpotent. This completes the proof. \square

We conclude this section with the following interesting information on the group M in the proof of Theorem 3.1. In fact, for the largest nilpotent group $M = \langle a, b \rangle$ relative to $a \in R_4(M)$, we have that M/T is isomorphic to the largest (nilpotent) 2-generated 4-Engel group $E(2, 4)$, where T is the torsion subgroup of M which is a $\{2, 3, 5\}$ -group. Therefore, in a nilpotent $\{2, 3, 5\}'$ -group, a right 4-Engel element with an arbitrary element generate a 4-Engel group. This can be seen by comparing the presentations of M/T and $E(2, 4)$ as follows. One can obtain two finitely presented groups $G1$ and $G2$ isomorphic to M/T and $E(2, 4)$, respectively by GAP:

```
MoverT:=FactorGroup(M,TorsionSubgroup(M));
E24:=NilpotentEngelQuotient(FreeGroup(2),4);
iso1:=IsomorphismFpGroup(MoverT);iso2:=IsomorphismFpGroup(E24);
G1:=Image(iso1);G2:=Image(iso2);
```

Next, we find the relators of the groups $G1$ and $G2$ which are two sets of relators on 13 generators by the following command in GAP.

```
r1:=RelatorsOfFpGroup(G1);r2:=RelatorsOfFpGroup(G2);
```

Now, save these two sets of relators by `LogTo` command of GAP in a file and go to the file to delete the terms as

```
<identity ...>
```

in the sets $r1$ and $r2$. Now call these two modified sets $R1$ and $R2$. We show that $R1=R2$ as two sets of elements of the free group f on 13 generators $f1, f2, \dots, f13$.

```
f:=FreeGroup(13);
f1:=f.1;f2:=f.2;f3:=f.3;f4:=f.4;f5:=f.5;f6:=f.6;
f7:=f.7;f8:=f.8;f9:=f.9;f10:=f.11;f12:=f.12;f13:=f.13;
```

Now by Read function, load the file in GAP and type the simple command $R1=R2$. This gives us `true` which shows G_1 and G_2 are two finitely presented groups with the same relators and generators and so they are isomorphic. We do not know if there is a guarantee that if someone else does as we did, then he/she finds the same relators for F_p groups G_1 and G_2 , as we have found. Also we remark that using function `IsomorphismGroups` to test if $G_1 \cong G_2$, did not give us a result in less than 10 hours and we do not know whether this function can give us a result or not.

We summarize the above discussion as following.

Theorem 3.4. *Let G be a nilpotent group generated by two elements, one of which is a right 4-Engel element. If G has no element of order 2, 3 or 5, then G is a 4-Engel group of class at most 6.*

4. Right n -Engel elements for $n \geq 5$

In this section we show that for every $n \geq 5$ there is a nilpotent group G of class $n+2$ containing elements a and $x \in R_n(G)$ such that both $[x^k, n a]$ and $[x^{-1}, n a]$ have infinite order for all integers $k \geq 2$.

Note that by Nickel's example [15], for every $n \geq 3$ we have already had a nilpotent group K of class $n+2$ containing a right n -Engel element x such that $[x^{-1}, n y] = [x^2, n y] \neq 1$ for some $y \in K$ i.e, neither x^2 nor x^{-1} are right n -Engel. We have checked by `nq` package of Nickel in GAP that $[x^{-1}, n y] = [x^2, n y]$ is of finite order whenever $n \in \{5, 6, 7, 8\}$. In fact,

- (1) $o([x^{-1}, 5 y]) = 3$, NqRuntime=1.7 Sec
- (2) $o([x^{-1}, 6 y]) = 7$, NqRuntime=54.8 Sec
- (3) $o([x^{-1}, 7 y]) = 4$, NqRuntime=1702 Sec
- (4) $o([x^{-1}, 8 y]) = 9$, NqRuntime=56406 Sec

Newman and Nickel [12] constructed a group H as follows. Let F be the relatively free group, generated by $\{a, b\}$ with nilpotency class $n+2$ and $\gamma_4(F)$ abelian. Let M be the (normal) subgroup of F generated by all commutators in a, b with at least 3 entries b and the commutators $[b, n+1 a]$ and $[b, n a, b]$. Then $H = \frac{F}{M}$. Note that the normal closure of b in H is nilpotent of class 2.

We denote the generators of H by a, b again. Put

$$t = [b, n a], \quad u_j = [b, n-1-j a, b, j a], \quad 0 \leq j \leq n-2,$$

$$u = \prod_{j=0}^{n-2} u_j, \quad v = [u_{n-2}, a], \quad w = \prod_{j=0}^{n-3} [u_j, a]$$

and let N be the subgroup $\langle tuw, t^2w, uw \rangle$. Then aN is a right n -Engel element in $\frac{H}{N}$ and $[b, n a]N$ has infinite order in $\frac{H}{N}$.

Now let H be the above group and $N_0 := \langle u, vw, vt^{-1} \rangle$. First, note that N_0 is a normal subgroup of H . For, clearly $t, v, w \in Z(H)$ and $u^b = u$. Also it is not hard

to see that $u_j^a = u_j[u_j, a]$ and thus $u^a = uvw$. This means that $N_0^a = N_0$ and so N_0 is a normal subgroup of H . Now we can state our main result of this section:

Theorem 4.1. $[b, n a]N_0 = [b^{-2}, n a]N_0$ and it has infinite order in $\frac{H}{N_0}$ and $[b^{-1}, n h] \in N_0$ for all $h \in H$. Furthermore $[b^{-k}, n a]N_0 = v^{(\frac{k}{2})}N_0$ for all $k \geq 2$.

Remark 4.2. As in [12], the proof of Theorem 4.1 involves a series of commutator calculations based, as usual, on the basic identities as following, which are mentioned in [12]. We bring them here for reader's convenience.

- (1) $[g, cd] = [g, d][g, c][g, c, d]$.
- (2) $[cd, g] = [c, g][c, g, d][d, g]$.
- (3) $[c^{-1}, d] = [c, d, c^{-1}]^{-1}[c, d]^{-1}$.
- (4) $[c, d^{-1}] = [c, d, d^{-1}]^{-1}[c, d]^{-1}$.
- (5) $[hk, h_1, \dots, h_s] = [h, h_1, \dots, h_s]$ for every k in $\gamma_{n+3-s}(H)$ and arbitrary $h_1, \dots, h_s \in H$.
- (6) $[g, d, c] = [g, c, d][g, [d, c]]k$, where k is a product of commutators of weight at least 4 with entries g, c and d .
- (7) $[a, n hk] = [a, n h]$ for all $h \in H$ and $k \in \gamma_3(H)$.
- (8) $[g, d^\delta] = [g, d]^\delta[g, 2d]^{(\frac{\delta}{2})}k$, where k is a product of commutators with at least 3 entries d and δ is positive.

Proof of Theorem 4.1. By Remark 4.2(7), we may assume that h is of the form $a^\alpha b^\beta [b, a]^\gamma$. The following calculations may depend to the signs of α and β ; we here outline only the case in which α and β are positive.

$$\begin{aligned} [b^{-1}, n h] &= [b^{-1}, n a^\alpha b^\beta [b, a]^\gamma] \\ &= [b^{-1}, n a^\alpha b^\beta] \prod_{j=0}^{n-1} [b^{-1}, n_{-1-j} a^\alpha b^\beta, [b, a]^\gamma, j a^\alpha b^\beta] \\ &= [b^{-1}, n a^\alpha b^\beta] ([b, [b, a], n_{-1} a] \prod_{j=0}^{n-2} [b, n_{-1-j} a, [b, a], j a])^{-\alpha^{n-1} \gamma}. \end{aligned}$$

Since

$$\begin{aligned} [b, [b, a], n_{-1} a] &= [[[b, a], b]^{-1}, n_{-1} a] \\ &= [b, a, b, n_{-1} a]^{-1} \\ &= v^{-1} \end{aligned}$$

and by Remark 4.2 (5) and (6)

$$[b, n_{-1-j} a, [b, a], j a] = [b, n_{-j} a, b, j a]^{-1} [b, n_{-1-j} a, b, j+1 a]$$

we have

$$\begin{aligned} \prod_{j=0}^{n-2} [b, n_{-1-j} a, [b, a], j a] &= \prod_{j=0}^{n-2} [b, n_{-j} a, b, j a]^{-1} [b, n_{-1-j} a, b, j+1 a] \\ &= \prod_{j=0}^{n-3} [b, n_{-1-j} a, b, j+1 a]^{-1} \prod_{j=0}^{n-2} [b, n_{-1-j} a, b, j+1 a] \\ &= v. \end{aligned}$$

Therefore

$$\begin{aligned} [b^{-1},_n a^\alpha b^\beta [b, a]^\gamma] &= [b^{-1},_n a^\alpha b^\beta] (v^{-1} v)^{-\alpha^{n-1} \gamma} \\ &= [b^{-1},_n a^\alpha b^\beta]. \end{aligned}$$

On the other hand by Remark 4.2 (8) we have

$$\begin{aligned} [b^{-1},_n a^\alpha b^\beta] &= [b^{-1},_n a^\alpha] \prod_{j=0}^{n-2} [b^{-1},_{n-1-j} a^\alpha, b^\beta,_{j+1} a^\alpha] \\ &= [b^{-1},_n a]^\alpha [b^{-1},_{n+1} a]^{n \binom{\alpha}{2} \alpha^{n-1}} \left(\prod_{j=0}^{n-2} [b,_{n-1-j} a, b,_{j+1} a] \right)^{-\alpha^{n-1} \beta} \\ &\quad \times \left(\prod_{j=0}^{n-3} [b,_{n-1-j} a, b,_{j+2} a] \right)^{-(n-2) \binom{\alpha}{2} \alpha^{n-2} \beta} \\ &\quad \times [b, a, b,_{n-2} a]^{-(n-2) \binom{\alpha}{2} \alpha^{n-2} \beta} [b, a, b]^{-(n-2) \binom{\alpha}{2} \alpha^{n-2} \beta} \\ &= (vt^{-1})^\alpha u^{-\alpha^{n-1} \beta} (vw)^{-(n-2) \binom{\alpha}{2} \alpha^{n-2} \beta}. \end{aligned}$$

Therefore $b^{-1}N_0$ is a right n -Engel element in $\frac{H}{N_0}$. This completes the second part of the theorem.

Since $\langle t, u, v, w \rangle$ is a free abelian group of rank 4, it is clear that $[b, a]N_0$ has infinite order. On the other hand

$$\begin{aligned} [b^{-2},_n a] &= [[b^{-1}, a][b^{-1}, a, b^{-1}][b^{-1}, a],_{n-1} a] \\ &= [b^{-1},_n a][b^{-1}, a, b^{-1},_{n-1} a][b^{-1}, a, b^{-1}, [b, a],_{n-2} a][b^{-1},_n a] \\ &\equiv [b, a, b,_{n-1} a] \pmod{N_0} \\ &\equiv v \pmod{N_0}. \end{aligned}$$

Since $vt^{-1} \in N_0$ we have $[b, a]N_0 = tN_0 = vN_0 = [b^{-2},_n a]N_0$. Now let $k \geq 2$, $f(1) = 0$ and $f(k) = (k-1) + f(k-1) = \binom{k}{2}$. Then

$$\begin{aligned} [b^{-k},_n a] &= [[b^{-1}, a][b^{-1}, a, b^{-(k-1)}][b^{-1}, a],_{n-1} a] \\ &= [b^{-1},_n a][b^{-1}, a, b^{-(k-1)},_{n-1} a][b^{-1}, a, b^{-(k-1)}, [b, a],_{n-2} a][b^{-(k-1)},_n a] \\ &\equiv [b, a, b^{(k-1)},_{n-1} a] v^{f(k-1)} \pmod{N_0} \\ &\equiv v^{f(k)} \pmod{N_0}. \end{aligned}$$

This completes the proof. \square

Now we answer negatively Question 1.1 which has been proposed in [1].

Let T be the torsion subgroup of H/N_0 and $x = bN_0T$ and $y = aN_0T$. Then the group $\mathcal{M} = H/N_0T = \langle x, y \rangle$ is a torsion free, nilpotent of class $n+2$, $x \in R_n(\mathcal{M})$ and both $[x^{-1},_n y]$ and $[x^k, a]$ are of infinite order for all integers $k \geq 2$. Since, for any given prime number p , a finitely generated torsion-free nilpotent group is residually finite p -group, it follows that for any prime number p and integer $k \geq 2$, there is a finite p -group $G(p, k)$ of class $n+2$ containing a right n -Engel element t such that both t^k and t^{-1} are not right n -Engel. This answers negatively Question 1.1.

5. Subgroupness of the set of (bounded) Left Engel elements of a group

Let $n = 2^k \geq 2^{48}$ and $B(X, n)$ be the free Burnside group on the set $X = \{x_i \mid i \in \mathbb{N}\}$ of the Burnside variety of exponent n defined by the law $x^n = 1$. Lemma 6 of [11] states that the subgroup $\langle x_{2k-1}^{n/2} x_{2k}^{n/2} \mid k = 1, 2, \dots \rangle$ of $B(X, n)$ is isomorphic to $B(X, n)$ under the map $x_{2k-1}^{n/2} x_{2k}^{n/2} \rightarrow x_k$, $k = 1, 2, \dots$. Therefore the subgroup $\mathcal{G} := \langle x_1^{n/2}, x_2^{n/2}, x_3^{n/2}, x_4^{n/2} \rangle$ is generated by four elements of order 2, contains the subgroup $\mathcal{H} = \langle x_1^{n/2} x_2^{n/2}, x_3^{n/2} x_4^{n/2} \rangle$ isomorphic to the free 2-generator Burnside group $B(2, n)$ of exponent n . One knows the tricky formulae

$$[x_k y] = [x, y]^{(-1)^{k-1} 2^{k-1}}$$

holding for all elements x and all elements y of order 2 in any group and all integers $k \geq 1$. It follows that the group \mathcal{G} can be generated by four left 49-Engel elements of \mathcal{G} . Thus

$$\mathcal{G} = \langle L_{49}(\langle \mathcal{G} \rangle) \rangle = \langle L(\langle \mathcal{G} \rangle) \rangle = \langle \overline{L}(\langle \mathcal{G} \rangle) \rangle,$$

where $L(H)$ ($\overline{L}(H)$, resp.) denotes the set of (bounded, resp.) left Engel elements of a group H .

Suppose, if possible, \mathcal{G} is an Engel group. Then \mathcal{H} is also an Engel group. Let Z and Y be two free generators of \mathcal{H} . Thus $[Z, k Y] = 1$ for some integer $k \geq 1$. Since \mathcal{H} is the free 2-generator Burnside group of exponent n , we have that every group of exponent n is a k -Engel group. Therefore, \mathcal{G} is an infinite finitely generated k -Engel group of exponent n , as \mathcal{H} is infinite by a celebrated result of Ivanov [10]. Hence, we have proved that

Proposition 5.1. *At least one of the following happens.*

- (1) *There is an infinite finitely generated k -Engel group of exponent n for some positive integer k and 2-power number n .*
- (2) *There is a group G such that $L(G) = \overline{L}(G)$ and $L(G)$ is not a subgroup of G .*

We believe that the subgroup \mathcal{H} cannot be an Engel group, but we are unable to prove it.

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